odic solution of the problem on oscillation of a shell with damping taken into account, under the conditions

$$u_i|_\Gamma = g_i(s, t), \quad i = 1, 2, 3, \quad \frac{\partial u_3}{\partial n}\Big|_\Gamma = g_4(s, t)$$

$$g_j(s, t+\omega) = g_j(s, t), \quad g_{jt}(s, t+\omega) = g_{jt}(s, t), \quad j = 1, 2, 3, 4$$

$$g_j(s, t) \in L^\infty(0, \omega), \quad g_{jt}(s, t) \in L^\infty(0, \omega), \quad j = 1, 2, 3, 4$$

$$g_j(s, t) \in H_{1/2}(\Gamma), \quad g_3(s, t) \in H_{3/2}(\Gamma), \quad j = 1, 2, 4$$

where $H_{1/2}(\Gamma)$ and $H_{3/2}(\Gamma)$ are the Sobolev-Slobodetskii spaces.

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PLANE SHORT-WAVE OSCILLATIONS IN THE VICINITY OF THE CONVEX BOUNDARY OF AN ELASTIC BODY

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We investigate short-wave oscillations of a plane elastic body, concentrated in the vicinity of a smooth convex boundary. We develop an asymptotic process of integrating the dynamic equations of the plane theory of elasticity. We obtain the expressions for the eigenfunctions and natural frequencies of the short-wave oscillations for free and clamped boundaries.

The short-wave (high frequency) oscillations can be studied with the help of various asymptotic methods based, in particular, on the method of rays of geometrical optics. A systematic presentation of the method of rays and its development in the boundary value problems of mathematical physics are given in [1, 2]. The method is used to investigate the asymptotic behavior of the eigenfunctions and eigenvalues of the Laplace operator for the case of large eigenvalues.

Use of the ray representations to describe the elastic high frequency oscillations was apparently first made in [3] in connection with the problem of reflection of a cylindrical wave from the boundary of a half-space. Authors of the later papers used the method of rays to solve various types of external problems of high frequency oscillations in elastic media. An extensive bibliography related to this problem is given in the survey [4]. There is, however, still no solution available to the problem of free high frequency oscillations of an elastic medium which fills a bounded region, when the oscillations penetrate the region to a certain depth. The general theoretical studies carried out in [5 — 7] also failed to supply sufficiently simple final results.

Below we present a generalization of the asymptotic method given in [2], to the solution of the fundamental internal dynamic problems of the plane theory of elasticity for the regions with smooth convex boundaries, in the case of free steady-state oscillations. The solution is developed for short-wave oscillations concentrated in a narrow strip in the vicinity of the boundary. The strip is contained between the boundary of the body and the caustic-curve behind which (in the inward direction) the oscillations decay exponentially. It was shown in [1] that the convexity of the boundary is a necessary condition for the appearance of such oscillations.

1. Let us assume that an isotropic homogeneous elastic medium fills a finite simply-connected region G bounded by a closed convex curve Γ. The coordinates of the points $x(s)$ and $y(s)$ on the boundary will be assumed differentiable with respect to the arc s, a sufficient number of times.

For the steady state oscillations, the components of the displacement vector $u(x,y)$ and $v(x,y)$ are expressed, in the case of plane deformation, in terms of the longitudinal and transverse potential $\varphi(x,y)$ and $\psi(x,y)$ as follows:

$$u = \frac{\partial\varphi}{\partial x} + \frac{\partial\psi}{\partial y}, \quad v = \frac{\partial\varphi}{\partial y} - \frac{\partial\psi}{\partial x}$$

The potentials φ and ψ satisfy the Helmholtz equations

$$\Delta\varphi + \frac{\omega^2}{c_1^2}\varphi = 0, \quad \Delta\psi + \frac{\omega^2}{c_2^2}\psi = 0 \qquad (1.1)$$

$$c_1^2 = \frac{\lambda + 2\mu}{\rho}, \quad c_2^2 = \frac{\mu}{\rho}, \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

where ρ, λ and μ are the density and the Lamé constants of the medium, and ω is the free-oscillation frequency.

We shall consider the short-wave oscillations near the boundary, using the intrinsic (s, n)-coordinate system where n is the normal to the boundary Γ and s is the arc length of Γ measured from the initial point. The outward direction of the normal is taken as positive, and the traversal of Γ is clockwise. Clearly, the (s, n)-coordinates are orthogonal and related to the Cartesian system as follows:

$$x = x(s) - ny'(s), \quad y = y(s) + nx'(s)$$

We choose the most common boundary conditions corresponding to the first and second fundamental boundary value problem of the plane theory of elasticity. These are the conditions for a clamped and a free boundary, and in the (s, n)-coordinate system they are expressed in terms of the potentials φ and ψ in the form

$$\frac{\partial\varphi}{\partial n} - \frac{\partial\psi}{\partial s}\bigg|_{n=0} = 0, \quad \frac{\partial\varphi}{\partial s} + \frac{\partial\psi}{\partial n}\bigg|_{n=0} = 0 \tag{1.2}$$

for a clamped boundary and

$$\frac{\partial^2\psi}{\partial n^2} - \frac{\partial^2\psi}{\partial s^2} + 2\frac{\partial^2\varphi}{\partial n\,\partial s} - \frac{1}{r}\left(2\frac{\partial\varphi}{\partial s} + \frac{\partial\psi}{\partial n}\right)\bigg|_{n=0} = 0 \tag{1.3}$$

$$\frac{\partial^2\varphi}{\partial n^2} - \frac{\partial^2\varphi}{\partial s^2} - 2\frac{\partial^2\psi}{\partial n\,\partial s} + \frac{1}{r}\left(2\frac{\partial\psi}{\partial s} - \frac{\partial\varphi}{\partial n}\right)\bigg|_{n=0} = 0$$

for a free boundary, where $r = r(s)$ is the radius of curvature of the boundary. Since the boundary is convex, the radius of curvature $r(s)$ is assumed positive everywhere.

2. The solutions of the Helmholtz equations constructed in [1, 2] correspond to the short-wave oscillations localized in a narrow boundary zone. The form of the solution of the Helmholtz equation in an arbitrary region was based on the analysis of the standard problem for a circle, and consists of an exponential term multiplied by the Airy function $\mathrm{Ai}(z)$.

We follow the example of [1, 2] and also use the problem for a circular region as standard. However, even in this simplest case the boundary conditions (1.2) or (1.3) cannot be separated into the conditions for φ and for ψ only. Consequently, Eqs. (1.1) form a fourth-order system. At the same time the standard problem allows the separation of variables. Let us set $\varphi = \Phi(r)e^{in\theta}, \quad \psi = \Psi(r)e^{in\theta}$

where (r, θ) denote the polar coordinates and $n \gg 1$ is the number of half-waves in in the peripheral direction. As the result of substituting (2.1) into (1.1), we arrive at the following system of Bessel equations:

$$\Phi'' + r^{-1}\Phi' + (\omega^2 c_1^{-2} - n^2 r^{-2})\,\Phi = 0 \tag{2.2}$$

$$\Psi'' + r^{-1}\Psi' + (\omega^2 c_2^{-2} - n^2 r^{-2})\,\Psi = 0$$

We seek a solution of (2.2) in a definite range of frequencies satisfying the condition

$$c_2 \leqslant \frac{\omega}{n} r < c_1 \tag{2.3}$$

It is in the range (2.3) of frequencies and only within this range that the oscillating integrals of Eqs. (2.2) are concentrated in a narrow strip near the boundary. In fact, the solution of the first equation of the system (2.2) under the condition (2.3) is expressed by an asymptotic expansion of the Bessel function $J_n(\omega r/c_1)$, when the argument and index are large values and their difference $(n - \omega r / c_1) \gg 1$. The solution of the second equation of (2.2) is expressed by an asymptotic expansion of the Bessel function $J_n(\omega r/c_2)$, when the argument is larger or equal to the index. The asymptotic formula for the function $J_n(\omega r / c_1)$ under the restriction indicated, has the form [8]

$$J_n(x) = \frac{1}{\pi}\sqrt{\frac{2(n-x)}{3x}}\; K_{1/3}(z), \quad x = \frac{\omega}{c_1} r, \quad z = \frac{[2(n-x)]^{3/2}}{3x^{1/2}} \tag{2.4}$$

$$K_{1/3}(z) = \left(\frac{\pi}{2z}\right)^{1/2} e^{-z}\left[1 - \frac{1-4/9}{8z} + \frac{(1-4/9)(9-4/9)}{128z^2} + \ldots\right]$$

From (2.4) it is clear that the integrals decay rapidly on moving away from the circular boundary. For the function $J_n(\omega r / c_2)$, we use the following uniform asymptotic expansion written in terms of the Airy function [1]:

$$J_n(x) = \frac{1}{\sqrt{\pi}}\left(\frac{2}{x}\right)^{1/3}\operatorname{Ai}(t), \quad t = (n-x)\left(\frac{2}{x}\right)^{1/3}, \quad x = \frac{\omega}{c_2}r \qquad (2.5)$$

The expansion (2.5) yields, in accordance with the properties of the Airy function, the integrals Ψ oscillating in the strip $r_* < r \leqslant R$ where R is the radius of the boundary and $r_* = nc_2 / \omega$, and decaying exponentially when $r < r_*$. The expansions (2.4) and (2.5) indicate that if we consider any frequency range different from (2.3), the integrals Φ and (or) Ψ either decrease uniformly within this range in an exponential manner beginning at the boundary, or oscillate at a considerable distance from the boundary.

Using the standard expansions (2.4) and (2.5) in a circular region, we represent the corresponding solutions of Eq. (1.1) for a region with an arbitrary convex boundary in the form

$$\varphi = Fe^{pf}, \quad \psi = \operatorname{Ai}(p^{2/3}\Psi)\, e^{ip\Phi}$$

where F, f, Ψ and Φ are unknown functions and p is an unknown large frequency parameter. We note that in the case of a circular boundary, the solutions (2.6) coincide with the asymptotic representations of the exact solutions.

3. Let us substitute the expressions (2.6) into the system (1.1). Dividing throughout by the exponential factors and taking into account the linear independence of the functions $\operatorname{Ai}(z)$ and $\operatorname{Ai}'(z)$, we obtain

$$p^2[\Psi(\nabla\Psi)^2 - (\nabla\Phi)^2] + ip\Delta\Phi + \omega^2 c_2^{-2} = 0 \qquad (3.1)$$

$$2ip^{5/3}(\nabla\Psi \cdot \nabla\Phi) + p^{2/3}\Delta\Psi = 0$$

$$p^2(\nabla f)^2 F + p[F\Delta f + 2(\nabla f \cdot \nabla F)] + \Delta F - \omega^2 c_1^{-2}F = 0$$

$$\Delta = a^{-1}\frac{\partial^2}{\partial s^2} + \frac{\partial^2}{\partial n^2} + a^{-1/2}\left[\frac{\partial}{\partial n}(a^{-1/2})\frac{\partial}{\partial n} + \frac{\partial}{\partial s}(a^{1/2})\frac{\partial}{\partial s}\right]$$

$$\nabla = a^{-1/2}\mathbf{e}_s\frac{\partial}{\partial s} + \mathbf{e}_n\frac{\partial}{\partial n}, \quad a = \left(1 + \frac{n}{r}\right)^2$$

We seek the unknown functions Ψ, Φ, F and the frequency ω in the form of expansions in inverse powers of the large frequency parameter

$$\Psi = \sum_0 \Psi_j p^{-j}, \quad \Phi = \sum_0 \Phi_j p^{-j}, \quad \omega^2 = p^2 + \sum_0 \varkappa_{-j+1} p^{-j+1} \qquad (3.2)$$

$$F = p^{-1/3}\sum_0 F_j^{(-1/3)}p^{-j} + p^{-2/3}\sum_0 F_j^{(-2/3)}p^{-j} + p^{-1}\sum_0 F_j^{(-1)}p^{-j}$$

The fact that F is represented as a combination of three different expansions in the fractional powers of p, follows from the requirement that the quantities used in the general boundary conditions must be of the same order of magnitude. Let us substitute the expansions (3.2) into Eqs. (3.1). Equating to zero the coefficients of like powers of p we obtain the following systems of recurrent equations:

$$\Psi_0 (\nabla \Psi_0)^2 - (\nabla \Phi_0)^2 + c_2^{-2} = 0, \quad (\nabla \Psi_0 \cdot \nabla \Phi_0) = 0 \tag{3.3}$$
$$\Psi_j (\nabla \Psi_0)^2 + 2\Psi_0 (\nabla \Psi_0 \cdot \nabla \Psi_j) - 2 (\nabla \Phi_0 \cdot \nabla \Phi_j) + c_2^{-2} \varkappa_{-j+2} = Q_{j-1}$$
$$(\nabla \Phi_0 \cdot \nabla \Psi_j) + (\nabla \Psi_0 \cdot \nabla \Phi_j) = N_{j-1} \quad (j = 1, 2, \ldots)$$
$$(\nabla f)^2 - c_1^{-2} = 0$$
$$2 (\nabla f \cdot \nabla F_0^{(-l/3)}) + F_0^{(-l/3)} (\Delta f - \varkappa_1 c_1^{-2}) = 0 \quad (l = 1, 2, 3)$$
$$(\nabla f \cdot \nabla F_j^{(-l/3)}) + F_j^{(-l/3)} \Delta f - c_1^{-2} \sum_{m=0}^{m=j} \varkappa_{j-m} F_{j-m}^{(-l/3)} = R_{j-1}$$

where the right-hand sides are fully determined by the solutions of the approximation equations of the lower order.

4. The boundary conditions for the functions Ψ, Φ, F and f follow directly from the conditions (1.2) and (1.3), by substituting the expressions (2.6) into them. Assuming that ψ and φ vary equally along the boundary, we obtain for $n = 0$

$$ip\Phi_{,n} \operatorname{Ai} + p^{2/3}\Psi_{,n} \operatorname{Ai}' + pf_{,s}F + F_{,s} = 0 \tag{4.1}$$
$$ip\Phi_{,s}\operatorname{Ai} + p^{2/3}\Psi_{,s}\operatorname{Ai}' - pf_{,n}F - F_{,n} = 0$$

for a clamped boundary and

$$\{p [\Psi (\nabla_* \Psi \cdot \nabla \Psi) - (\nabla_* \Phi \cdot \nabla \Phi)] + i\Delta_* \Phi\} \operatorname{Ai} + \tag{4.2}$$
$$p^{2/3} [2i (\nabla_* \Phi \cdot \nabla \Psi) + p^{-1} \Delta_* \Psi] \operatorname{Ai}' + [p (\nabla_1 f \cdot \nabla f) + \Delta_1 f] F +$$
$$2 (\nabla_1 f \nabla F) + p^{-1} \Delta_1 F = 0$$
$$\{p [(\nabla_1 \Phi \cdot \nabla \Phi) - \Psi (\nabla_1 \Psi \cdot \nabla \Psi)] - i\Delta_1 \Phi\} \operatorname{Ai} - p^{2/3} [2i \times$$
$$(\nabla_1 \Psi \cdot \nabla \Phi) + p^{-1} \Delta_1 \Psi] \operatorname{Ai}' + [p (\nabla_* f \cdot \nabla f) + \Delta_* f] F +$$
$$2 (\nabla_* f \cdot \nabla F) + p^{-1} \Delta_* F = 0$$
$$\nabla_* = \mathbf{e}_n \frac{\partial}{\partial n} - \mathbf{e}_s \frac{\partial}{\partial s}, \quad \nabla_1 = \mathbf{e}_n \frac{\partial}{\partial s} + \mathbf{e}_s \frac{\partial}{\partial n}$$
$$\Delta_* = \frac{\partial^2}{\partial n^2} - \frac{\partial^2}{\partial s^2} - \frac{1}{r} \frac{\partial}{\partial n}, \quad \Delta_1 = 2 \left(\frac{\partial^2}{\partial n \partial s} - \frac{1}{r} \frac{\partial}{\partial s} \right)$$

for a free boundary. Here a comma preceding n or s denotes a partial derivative with respect to the corresponding coordinate.

The conditions obtained contain the functions Ai and Ai′. To construct the system of boundary conditions corresponding to the systems of recurrent equations (3.3), we write Ai and Ai′ in the form of expansions in powers of the difference $p^{2/3} (\Psi - \Psi_0)$

$$\operatorname{Ai} (p^{2/3}\Psi) = \operatorname{Ai} (p^{2/3}\Psi_0) + \operatorname{Ai}' (p^{2/3}\Psi_0)(p^{-1/3}\Psi_1 + \ldots) + \ldots \tag{4.3}$$
$$\operatorname{Ai}' (p^{2/3}\Psi) = \operatorname{Ai}' (p^{2/3}\Psi_0) + \operatorname{Ai}'' (p^{2/3}\Psi_0)(p^{-1/3}\Psi_1 + \ldots) + \ldots$$

Replacing the unknown functions, the frequency and the Airy functions in the conditions (4.1) and (4.2) by the expansions (3.2) and (4.3) and equating to zero the coefficients of like powers of p, we arrive at the system of the boundary conditions for the consecutive approximations.

Let us first write the explicit expressions for the first order approximations to the boundary conditions, which are the same for both problems and have the form

$$\mathrm{Ai}\,(p^{2/3}\Psi_0)\,|_{n=0} = 0$$

from which it follows at once that

$$\Psi_0\,|_{\,n=0} = p^{-2/3}t_q \tag{4.4}$$

where t_q, $(q = 1, 2, \ldots)$ is one of the first roots of the Airy function. Moreover, the requirement that the solutions be periodic and vary along the boundary in an identical manner, leads to the following conditions (for $n = 0$) :

$$i\Phi = f, \quad [p\Phi] = 2\pi M \tag{4.5}$$

where $M \gg 1$ is an integer, and the square brackets denote the increment in the value of the function in question on traversing the boundary contour once.

5. We now construct the solutions for the recurrent systems of equations. Since the solutions are sought in a narrow strip $|\,n\,| < 1$, we can write the functions Ψ_j, Φ_j, $F_j^{(-l/3)}$, f and the coefficient a in the form of Taylor series in terms of the coordinate n

$$\Psi_j = \sum_0 \Psi_{jk}n^k, \quad \Phi_j = \sum_0 \Phi_{jk}n^k, \quad f = \sum_0 f_k n^k \tag{5.1}$$
$$F_j^{(-l/3)} = \sum_0 F_{jk}^{(-l/3)}n^k, \quad a = 1 + 2a_1 n + a_1^2 n^2, \quad a_1 = r^{-1}$$

Substituting the expansions (5.1) into the system of differential equations (3.3), we can reduce the latter to a system of algebraic equations for the coefficients appearing in the expansions for the unknown functions and their derivatives with respect to n.

Let us consider the solution of the first two approximation equations which are sufficient for determining the frequencies of the natural oscillations with the accuracy of $O\,(p^{-1})$. The first approximation equations are

$$f_{,n}^2 + a^{-1}f_{,s}^2 - c_1^{-2} = 0, \quad \Phi_{0,n}\Psi_{0,n}' + a^{-1}\Phi_{0,s}\Psi_{0,s} = 0 \tag{5.2}$$
$$\Psi_0\,(\Psi_{0,n}^2 + a^{-1}\,\Psi_{0,s}^2) - \Phi_{0,n}^2 - a^{-1}\,\Phi_{0,s}^2 + c_2^{-2} = 0$$

Let us replace the functions Ψ_0, Φ_0 and f by their corresponding expansions from (5.1) and extract from the resulting recurrent system, the equations for the terms proportional to n^k $(k = 0, 1, 2)$, omitting those which are not used in the process of determining the frequency with the required accuracy

$$f_1^2 + f_{0,\,s}^2 - c_1^{-2} = 0, \quad \varepsilon\Psi_{01}^2 - \Phi_{00,\,s}^2 + c_2^{-2} = 0, \quad \Phi_{01} = 0 \tag{5.3}$$
$$\Psi_{01}^3 + 2a_1c_2^{-2} + 2\varepsilon\Psi_{01}(a_1\Psi_{01} + \Psi_{02}) = 0, \quad \Phi_{00,\,s}\Psi_{01,\,s} + 2\Phi_{02}\Psi_{01} = 0$$
$$\Psi_{01}^2\,(5\Psi_{02} + 2a_1\Psi_{01}) - 4\Phi_{02}^2 - 2\Phi_{00,\,s}\Phi_{02,\,s} + a_1^2c_2^{-2} + \varepsilon\,[\Psi_{01}\,(a_1^2\Psi_{01} + 8a_1\Psi_{02} + 6\Psi_{03}) + 4\Psi_{02}^2 + \Psi_{01,\,s}^2] = 0$$
$$\varepsilon = t_q p^{-2/3}$$

It is clear from (5.3) that the coefficients of the expansions of the unknown functions (5.1) depend on the small parameter ε. Let us write them in the form of the following asymptotic sums:

$$\Phi_{jk} = \sum_0 \Phi_{jkm}\varepsilon^m, \quad \Psi_{jk} = \sum_0 \Psi_{jkm}\varepsilon^m, \quad \Psi_{00} = \varepsilon \tag{5.4}$$

$$f_k = \sum_0 f_{km}\varepsilon^m, \quad F_{jk}^{(-l/3)} = \sum_0 F_{jkm}^{(-l/3)}\varepsilon^m$$
$$(j, k = 0, 1, 2, \ldots; l = 1, 2, 3)$$

Substituting the sums (5.4) into Eqs. (5.3) and equating the coefficients of like-powers of ε, we obtain a new system of recurrent equations. Below we give the necessary equations of the three approximations in ε for the terms proportional to n^0, equations of two approximations in ε for the terms proportional to n^1 and the first approximation equations in ε for the terms proportional to n^2:

$$\Phi_{000,s}{}^2 - c_2^{-2} = 0, \quad f_{10}{}^2 + f_{00,s}{}^2 - c_1^{-2} = 0, \quad \Psi_{010}{}^3 + 2a_1c_2^{-2} = 0 \qquad (5.5)$$
$$\Psi_{010}(2a_1 + 3\Psi_{011}) + 4\Psi_{020} = 0, \quad 2\Phi_{020}\Psi_{010} + c_2^{-1}\Psi_{010,s} = 0$$
$$2\Psi_{010}\Psi_{011} - 2c_2^{-1}\Phi_{002,s} - \Phi_{001,s}{}^2 = 0, \quad \Psi_{010}{}^2 - 2c_2^{-1}\Phi_{001,s} = 0$$
$$\Psi_{010}{}^2(2a_1\Psi_{010} + 5\Psi_{020}) - 4\Phi_{020}{}^2 - 2c_2^{-1}\Phi_{020,s} = 0$$

Solving the algebraic system (5.5) directly, we find the following approximate expressions for the unknown functions:

$$\Phi_{000} = \pm c_2^{-1}s, \quad \Psi_{020} = {}^1/_5(2c_2)^{-2/3}r^{-4/3}[{}^4/_9(r')^2 - rr'' + 4] \qquad (5.6)$$
$$\Phi_{001} = (2c_2)^{-1/3}\int_0^s r^{-2/3}(\tau)\,d\tau, \quad \Psi_{010} = -r^{-1/3}\left(\frac{2}{c_2^2}\right)^{1/3}$$
$$\Psi_{011} = \frac{2}{15}\left[\frac{4}{9}\frac{(r')^2}{r} - \frac{1}{3}r'' - \frac{1}{r}\right], \quad \Phi_{020} = \frac{r'}{6c_2r}$$
$$\Phi_{002} = \frac{1}{15}\left(\frac{c_2}{4}\right)^{1/3}\int_0^s r^{-4/3}(\tau)\left[\frac{4}{3}r(\tau)r''(\tau) - \frac{16}{3}(r'(\tau))^2 - \frac{7}{2}\right]$$

The function $f_{00}(s)$ is found from the condition that the solutions vary along the boundary in an identical manner: $f_{00} = i\,\Phi_{000}$. Taking into account (5.6), we obtain

$$f_{00} = \pm ic_2^{-1}s, \quad f_{10} = (c_1^{-2} + c_2^{-2})^{1/2} \qquad (5.7)$$

The plus sign in f_{10} is chosen so as to ensure a rapid decay of the integrals φ on moving away from the boundary. The frequency parameter p is found from the first approximation of the conditions of periodicity of (4.5)

$$[p\Phi_0] = 2\pi M, \quad M \gg 1$$

or in its expanded form

$$p\int_0^L \Phi_{000,s}ds + t_qp^{1/3}\int_0^L \Phi_{001,s}ds + t_q^2p^{-1/3}\int_0^L \Phi_{002,s}ds + \ldots = 2\pi M \qquad (5.8)$$

where L is the length of the boundary contour, $q = 1, 2, \ldots$ and M is an integer. Substituting the expressions (5.6) for Φ_{00m} $(m = 0, 1, 2)$ into (5.8) and solving the resulting equation by the method of consecutive approximations in p, we find

$$p = M(b_q^{(0)} + b_q^{(1)}M^{-2/3} + b_q^{(2)}M^{-4/3} + \ldots) \qquad (5.9)$$
$$M \gg 1, \quad q = 1, 2, \ldots$$
$$b_q^{(0)} = 2\pi e_1, \quad b_q^{(1)} = e_1e_2^{1/3}t_q\int_0^L r^{-2/3}(s)\,ds$$

$$b_q^{(2)} = \frac{7}{60} e_1 e_2^{-1/3} t_q^2 \int_0^L r^{-4/3}(s)\, ds, \quad e_1 = \frac{c_2}{L}, \quad e_2 = \frac{\pi}{L}$$

The above relations show that the magnitude of the frequency parameter and hence of the first approximation for the natural frequency oscillations, are independent of the boundary conditions type.

Using the expressions given in (5.6) for the functions Ψ_{00} and Ψ_{01}, we now determine the zone of oscillations of the integrals ψ. The annular zone of oscillation of the integrals (2.6) is, in accordance with the properties of the Airy function $\mathrm{Ai}\,(t)$, contained between the boundary of the region in which $t = t_q$, and the caustic curve on which the argument of the Airy function becomes equal to zero $(t = 0)$. Since in the present case $t = p^{2/3}\Psi$, the condition that $\Psi = 0$ and the expansions (3.2), (5.1) together yield the following equation of the caustic curve in its first approximation:

$$n_* = -\Psi_{00}\Psi_{01}^{-1} \approx {}^1/_2\, t_q r^{1/3} (e_2 M)^{-2/3} \tag{5.10}$$

Consequently the zone of oscillations satisfies the inequalities

$$n_* \leqslant n < 0 \tag{5.11}$$

From (5.11) follows the condition of convexity of the boundary: $r\,(s) > 0$, since the zone of oscillations is contained within the region $n < 0$ and $t_q < 0$ within this zone. The width of the oscillation zone is of the order $O\,(M^{-2/3})$. The latter result agrees with those given in [1, 2].

The second approximation equations are

$$\begin{gathered} 2\Psi_0 (\nabla\Psi_0 \cdot \nabla\Psi_1) + \Psi_1 (\nabla\Psi_0)^2 - 2(\nabla\Phi_0 \cdot \nabla\Phi_1) + \\ \varkappa_1 c_2^{-2} = -i\Delta\Phi_0 \\ 2i\,[(\nabla\Phi_1 \cdot \nabla\Psi_0) + (\nabla\Phi_0 \cdot \nabla\Psi_1)] = -\Delta\Psi_0 \end{gathered} \tag{5.12}$$

The unknown parameter $\varkappa_1$ determining the second approximation for the natural frequency, enters the first equation of (5.12). Let us substitute into this equation the expansions (5.1) and (5.4), and separate the terms proportional to the zero powers of n and ε. This yields

$$2\Phi_{100,\,s} - \Psi_{100}\Psi_{010}^2 - \varkappa_1 c_2^{-2} = 2i\Phi_{020} \tag{5.13}$$

To find the parameter $\varkappa_1$ we use the second approximation to the conditions of periodicity (4.5): $[\Phi_1] = 0$. Substituting into (5.13) the known expression for Φ_{020} and performing the integration with the periodicity of the radius of curvature of the boundary contour $(r\,(s + L) = r\,(s))$ taken into account, we obtain

$$\varkappa_1 = -c_2 e_1 \int_0^L \Psi_{100}(s)\, \Psi_{010}^2(s)\, ds \tag{5.14}$$

We find the unknown function $\Psi_{100}\,(s)$ from the second approximation of the boundary conditions (4.1) and (4.2). These conditions, unlike in the first approximation, no longer coincide in the case of the free and clamped boundaries. Substituting into (4.1) and (4.2) the expansions (5.4), we obtain the following boundary conditions for the coefficients proportional to ε°:

$$\begin{gathered} f_{10} F_{000}^{(-1/3)} - i\Phi_{000,\,s}\Psi_{100}\,\mathrm{Ai}'(t_q) = 0 \\ f_{00,\,s} F_{000}^{(-1/3)} + \Psi_{010}\,\mathrm{Ai}'(t_q) = 0 \end{gathered} \tag{5.15}$$

for the clamped boundary and

$$2f_{00,\,s}f_{10}F_{000}{}^{(-1/3)} + \Phi_{000,\,s}{}^{2}\Psi_{100}\,\mathrm{Ai}'(t_q) = 0$$
$$(f_{10}{}^{2} - f_{00,\,s}{}^{2})\,F_{000}{}^{(-1/3)} - 2i\Phi_{000,\,s}\Psi_{010}\,\mathrm{Ai}'(t_q) = 0 \quad (5.16)$$

for a free edge.

Considering now (5.15) and (5.16) as a system of two equations with two unknowns $F_{000}^{(-1/3)}$ and Ψ_{100}, we find

$$F_{000}{}^{(-1/3)} = -ie^{1/3}\,\mathrm{Ai}'(t_q),\ \Psi_{100} = -e^{1/3}(1 + e_3{}^{2})^{1/2} \quad (5.17)$$
$$e = 2c_2/r,\ e_3 = c_2/c_1$$

for a clamped boundary and

$$F_{000}{}^{(-1/3)} = -2ie^{1/3}(2 + e_3{}^{2})^{-1}\,\mathrm{Ai}'(t_q), \quad \Psi_{100} = -4e^{1/3}(2 + e_3{}^{2})^{-1}(1 + e_3{}^{2})^{1/2} \quad (5.18)$$

for a free edge.

Substituting now (5.17) and (5.18) into (5.14), we find that the parameter $\varkappa_1$ for the regions with a clamped $(\varkappa_{1(*)})$ and a free $(\varkappa_{1(0)})$ boundary is equal to

$$\varkappa_{1(*)} = 2e_1(1 + e_3{}^{2})^{1/2}\int_0^L r^{-1}(s)\,ds$$

$$\varkappa_{1(0)} = 8e_1(1 + e_3{}^{2})^{1/2}(2 + e_3{}^{2})^{-1}\int_0^L r^{-1}(s)\,ds$$

respectively.

Thus we have obtained for both problems the second approximation for the natural frequency oscillations, and this enables us to determine this frequency sufficiently accurately, with the error of $O(p^{-1})$.

The third and further approximations are constructed in the similar manner. We use the general systems (3.3) to extract the equations for the next approximation and the corresponding boundary conditions. Substitution of (5.1) into these equations reduces them to a linear algebraic system of complexity increasing directly with the order of the approximation, and consequently more and more difficult to solve.

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INVERSION OF RELATIONS OF THE THEORY OF PLASTIC FLOW FOR HARDENING BODIES

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The inversion of the fundamental relationships of the theory of plastic flow of hardening bodies is obtained in the neighborhood of a regular point of an arbitrary loading surface. The stress increments are consequently expressed explicitly in terms of the strain increments.

The fundamental relationships of the theory of a plastic hardening body [1, 2] under the assumption of the existence of loading functions are in the form of relationships expressing the increments of strain in terms of the increments of stress. Upon formulating the problems in displacements, for example in the case of three-dimensional stability problems [3, 4], the increments in stress must be expressed in terms of the increments of strain, i. e. the fundamental relationships must be inverted. Such an inversion is realized below in the neighborhood of a regular point of an arbitrary loading surface for an isothermal strain process in the case of small strains.

1. Following [1, 2], let us write the fundamental relationships of the theory of a plastic hardening body in the neighborhood of a regular point of the loading surface. We represent the total strain increment as the sum of increments in the elastic and plastic strains (we introduce the compliance tensor C for the elastic strain, and we proceed from the associated flow law for plastic deformation)

$$d\varepsilon_{nm} = d\varepsilon_{nm}{}^e + d\varepsilon_{nm}{}^p \tag{1.1}$$

$$d\varepsilon_{nm}^e = C_{nmij}\, d\sigma^{ij} \tag{1.2}$$

$$d\varepsilon_{nm}^p = d\lambda \frac{\partial f}{\partial \sigma^{mn}}, \quad \text{when} \quad f = 0,\ df = 0 \quad \text{and} \quad d'f > 0 \tag{1.3}$$

$$d\varepsilon_{nm}^p = 0, \ \text{when} \ \ f = 0 \quad \text{and} \quad df \equiv d'f \leqslant 0 \quad \text{or} \quad f < 0$$

where f denotes the loading function; the equation of the loading surface is hence

$$f(\sigma^{ij}, g_{ij}, \varepsilon_{ij}{}^p, \chi_s, k_s) = 0 \tag{1.4}$$